

THERMAL STRESSES IN CHIRAL PLATES

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This paper is concerned with the linear theory of chiral Cosserat thermoelastic bodies. We investigate the deformation of chiral plates. First, we present the basic equations which govern the deformation of thin plates. Then, we present reciprocity and uniqueness results. In the next section we establish the instability of solutions whenever the internal energy is negative. We use a semigroup approach to prove the existence of solution. The deformation of an infinite plate with a circular hole is investigated.

Keywords: Thermal stresses in plates; Chiral materials; Uniqueness results.

INTRODUCTION

The behavior of chiral materials is of interest for the investigation of carbon nanotubes, auxetic materials, bones, honeycomb structures, as well as composites with inclusions. The deformation of chiral elastic materials cannot be described within classical elasticity [1,2]. Various authors have studied the behavior of chiral materials by using the theory of Cosserat elasticity (see, e.g., [3-10] and references therein). The Cosserat theory studies continua with oriented particles, in which each material point has the six degrees of freedom of a rigid body. The basic equations of the linear theory of Cosserat thermoelasticity have been studied in various books (see, e.g., Nowacki [11], Eringen [12], Dyszlewicz [13]). The deformation of achiral Cosserat elastic plates has been investigated by various authors. A detailed analysis of the results established in

this theory have been presented in [14].

In this paper we use the results of Eringen [15], Nowacki [16] and Inan[17] to derive a theory of isotropic chiral plates. We assume that on the upper and lower faces of the plate there are prescribed the surface traction, the surface moment and the heat flux. We show that, in contrast with the theory of achiral plates, the stretching and flexure cannot be treated independently of each other. The paper is structured as follows. First, we present the basic equations of homogenous and isotropic chiral Cosserat thermoelastic solids. Then, we establish a theory of thermoelastic thin plates. In the following section we present reciprocity and uniqueness results in the dynamic theory. Then, by means of the logarithmic convexity arguments we investigate the instability of solutions. We use a semigroup approach to establish an existence result. Finally, the deformation of an infinite plate with a circular hole is investigated. It is shown that a constant thermal field in a chiral plate produces a bending effect.

BASIC EQUATIONS

In this section we present the basic equations of the thermoelasticity for isotropic chiral Cosserat continua. We refer the motion of the continuum to a fixed system of rectangular Cartesian axes Ox_k , ($k = 1, 2, 3$). We consider a body that at time t_0 occupies the regular region B of Euclidean three-dimensional space and is bounded by the surface ∂B . We designate by \mathbf{n} the outward unit normal of ∂B . Letters in boldface stand for tensors of an order $p \geq 1$, and if \mathbf{v} has the order p , we write $v_{ij\dots s}$ (p subscripts) for the components of \mathbf{v} in the Cartesian coordinate system. We shall employ the usual

summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$ whereas Greek subscripts to the range $(1, 2)$; summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

We assume that B is occupied by a homogeneous and isotropic chiral Cosserat thermoelastic material. We denote by u_i the components of the displacement vector and by φ_i the components of the microrotation vector. The strain measures in the linear theory are defined by

$$e_{ij} = u_{j,i} + \varepsilon_{ijk}\varphi_k \quad \kappa_{ij} = \varphi_{j,i} \quad (1)$$

where ε_{ijk} is the alternating symbol. Let t_{ij} be the stress tensor and let m_{ij} be the couple stress tensor over B . The equations of motion of a Cosserat continua can be expressed as

$$\begin{aligned} t_{ji,j} + f_i &= \rho \ddot{u}_i \\ m_{ji,j} + \varepsilon_{ijk} t_{jk} + g_i &= J \ddot{\varphi}_i \end{aligned} \quad (2)$$

where f_i is the body force, g_i is the body couple, ρ is the reference mass density, and J is a coefficient of inertia. The energy equation is given by

$$\rho T_0 \dot{\eta} = q_{i,i} + s \quad (3)$$

where η is the entropy, q_i is the heat flux vector, s is the heat supply, and T_0 is the constant absolute temperature of the body in the reference configuration.

In the context of the linear theory of isotropic chiral and homogeneous thermoelastic bodies the constitutive equations have the form [1, 12, 13]

$$\begin{aligned}
t_{ij} &= \lambda e_{rr} \delta_{ij} + (\mu + \kappa) e_{ij} + \mu e_{ji} + C_1 \kappa_{ss} \delta_{ij} + C_2 \kappa_{ji} + C_3 \kappa_{ij} - b_1 \theta \delta_{ij} \\
m_{ij} &= \alpha \kappa_{rr} \delta_{ij} + \beta \kappa_{ji} + \gamma \kappa_{ij} + C_1 e_{rr} \delta_{ij} + C_2 e_{ji} + C_3 e_{ij} - b_2 \theta \delta_{ij} \\
\rho \eta &= b_1 e_{rr} + b_2 \kappa_{rr} + a \theta \quad q_i = k \theta_{,i}
\end{aligned} \tag{4}$$

where δ_{ij} is the Kronecker delta, θ is the temperature measured from the constant absolute temperature of the reference state, and $\lambda, \mu, \kappa, \alpha, \beta, \gamma, C_1, C_2, C_3, b_1, b_2, a$ and k are constitutive constants. In the case of an achiral material the coefficients C_1, C_2, C_3 and b_2 are equal to zero.

The components of surface traction, the components of surface couple and the heat flux at regular points of ∂B are defined by

$$t_i = t_{ji} n_j \quad m_i = m_{ji} n_j \quad q = q_i n_i \tag{5}$$

respectively. Let $P = (u_i, \varphi_i, \theta)$. We say that P is an admissible process on $B \times \mathcal{T}$ provided: (i) u_i and φ_i are of class C^2 on $B \times \mathcal{T}$; (ii) θ is of class $C^{2,1}$ on $B \times \mathcal{T}$; (iii) u_i, φ_i and θ are of class C^1 on $\overline{B} \times \mathcal{T}$. We assume that f_i, g_i and s are continuous on $B \times \mathcal{T}$ and that ρ and J are positive constants.

THERMOELASTIC PLATES

In what follows we assume that the region B refers to the interior of a right cylinder of length $2h$ with open cross-section Σ and the smooth lateral boundary Π . Let Γ be

the boundary of Σ . The rectangular Cartesian coordinate frame consists of the origin O and the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The coordinate frame is supposed to be chosen in such a way that the plane x_1Ox_2 is middle plane. Thus, we have

$$B = \{x : (x_1, x_2) \in \Sigma, -h < x_3 < h\}, \quad \Pi = \{x : (x_1, x_2) \in \Gamma, -h < x_3 < h\}$$

We derive a theory of thin plates of uniform thickness where the displacements, the microrotations and the temperature have the form

$$\begin{aligned} u_\alpha &= w_\alpha(x_1, x_2, t) + x_3 v_\alpha(x_1, x_2, t) \quad u_3 = w_3(x_1, x_2, t) \\ \varphi_j &= \Phi_j(x_1, x_2, t) \\ \theta &= T_1(x_1, x_2, t) + x_3 T_2(x_1, x_2, t) \quad (x_1, x_2, x_3) \in B \quad t \in \mathcal{T} \end{aligned} \tag{6}$$

Following [15-17], to establish a plate theory we perform the following integrations: (i) we integrate equations of balance of momenta with respect to x_3 over the thickness of the plate; (ii) we take the cross product of the equations of the balance of linear momentum with $x_3 \mathbf{e}_3$ and integrate over the thickness of the plate; (iii) we integrate the equation of energy over x_3 between the limits $-h$ and h ; (iv) we multiply the equation of energy by x_3 and integrate over the thickness of the plate. The results of (i) are

$$\tau_{\beta k, \beta} + F_k = \rho \ddot{w}_k \quad \mu_{\beta \alpha, \beta} + \varepsilon_{3\rho\alpha}(\tau_{3\rho} - \tau_{\rho 3}) + G_\alpha = J \ddot{\Phi}_\alpha \quad \mu_{\alpha 3, \alpha} + \varepsilon_{3\rho\alpha} \tau_{\rho\alpha} + G_3 = J \ddot{\Phi}_3 \tag{7}$$

where

$$\begin{aligned} \tau_{ij} &= \frac{1}{2h} \int_{-h}^h t_{ij} dx_3 \quad \mu_{ij} = \frac{1}{2h} \int_{-h}^h m_{ij} dx_3 \\ F_i &= \frac{1}{2h} \int_{-h}^h f_i dx_3 + \frac{1}{2h} [t_{3i}]_{-h}^h \quad G_i = \frac{1}{2h} \int_{-h}^h g_i dx_3 + \frac{1}{2h} [m_{3i}]_{-h}^h \end{aligned} \tag{8}$$

We assume that the functions t_i, m_i and q are prescribed on the surfaces $x_3 = \pm h$.

To the equations (6) we add the result of (ii), i.e.,

$$M_{\beta\alpha,\beta} - 2h\tau_{3\alpha} + H_\alpha = \rho I \ddot{v}_\alpha \quad (9)$$

where we have used the notations

$$M_{\alpha\beta} = \int_{-h}^h x_3 t_{\alpha\beta} dx_3 \quad I = \frac{2}{3}h^3 \quad H_\alpha = \int_{-h}^h x_3 f_\alpha dx_3 + [x_3 t_{3\alpha}]_{-h}^h \quad (10)$$

If we integrate the equation (3) with respect to x_3 between the limits $-h$ and h , then we obtain the following equation

$$\rho T_0 \dot{\zeta} = \chi_{\alpha,\alpha} + S_1 \quad (11)$$

where the functions ζ, χ_j and S_1 are defined by

$$\begin{aligned} \zeta &= \frac{1}{2h} \int_{-h}^h \eta dx_3 \quad \chi_j = \frac{1}{2h} \int_{-h}^h q_j dx_3 \\ S_1 &= \frac{1}{2h} \int_{-h}^h s dx_3 + \frac{1}{2h} [q_3]_{-h}^h \end{aligned} \quad (12)$$

The equation which results from the multiplication of Eq. (3) by x_3 and integration over x_3 from $x_3 = -h$ to $x_3 = h$ can be written in the form

$$\rho T_0 \dot{\sigma} = Q_{\alpha,\alpha} - 2h\chi_3 + S_2 \quad (13)$$

where we have used the notations

$$\sigma = \int_{-h}^h x_3 \eta dx_3 \quad Q_\alpha = \int_{-h}^h x_3 q_\alpha dx_3 \quad S_2 = \int_{-h}^h x_3 s dx_3 + [x_3 q_3]_{-h}^h \quad (14)$$

The functions F_j, G_j, H_α and S_α are prescribed. From (1) and (6) we obtain

$$e_{\alpha\beta} = \gamma_{\alpha\beta} + x_3 \xi_{\alpha\beta} \quad e_{33} = 0 \quad e_{\alpha 3} = \gamma_{\alpha 3} \quad e_{3\alpha} = \gamma_{3\alpha} \quad \kappa_{\alpha j} = \eta_{\alpha j} \quad \kappa_{3j} = 0 \quad (15)$$

where

$$\gamma_{\alpha j} = w_{j,\alpha} + \varepsilon_{j\alpha k} \Phi_k \quad \gamma_{3\alpha} = v_\alpha + \varepsilon_{3\beta\alpha} \Phi_\beta \quad \eta_{\alpha k} = \Phi_{k,\alpha} \quad \xi_{\alpha\beta} = v_{\beta,\alpha} \quad (16)$$

It follows from (4), (8), (10), (12), (14), (15) and (6) that

$$\begin{aligned} \tau_{\alpha\beta} &= \lambda \gamma_{\rho\rho} \delta_{\alpha\beta} + (\mu + \kappa) \gamma_{\alpha\beta} + \mu \gamma_{\beta\alpha} + C_1 \eta_{\rho\rho} \delta_{\alpha\beta} \\ &\quad + C_2 \eta_{\beta\alpha} + C_3 \eta_{\alpha\beta} - b_1 T_1 \delta_{\alpha\beta} \\ \tau_{\alpha 3} &= (\mu + \kappa) \gamma_{\alpha 3} + \mu \gamma_{3\alpha} + C_3 \eta_{\alpha 3} \\ \tau_{3\alpha} &= (\mu + \kappa) \gamma_{3\alpha} + \mu \gamma_{\alpha 3} + C_2 \eta_{\alpha 3} \\ \mu_{\nu\kappa} &= \alpha \eta_{\rho\rho} \delta_{\nu\kappa} + \beta \eta_{\kappa\nu} + \gamma \eta_{\nu\kappa} + C_1 \gamma_{\rho\rho} \delta_{\nu\kappa} + C_2 \gamma_{\kappa\nu} + C_3 \gamma_{\nu\kappa} - b_2 T_1 \delta_{\nu\kappa} \\ \mu_{\alpha 3} &= \beta \gamma_{3\alpha} + \gamma \eta_{\alpha 3} + C_2 \gamma_{3\alpha} + C_3 \gamma_{\alpha 3} \\ M_{\alpha\beta} &= I[\lambda \xi_{\rho\rho} \delta_{\alpha\beta} + (\mu + \kappa) \xi_{\alpha\beta} + \mu \xi_{\beta\alpha} - b_1 T_2 \delta_{\alpha\beta}] \\ \rho \zeta &= b_1 \gamma_{\rho\rho} + b_2 \eta_{\rho\rho} + a T_1 \quad \chi_\alpha = k T_{1,\alpha} \quad \chi_3 = k T_2 \\ \rho \sigma &= I(b_1 \xi_{\rho\rho} + a T_2) \quad Q_\alpha = k I T_{2,\alpha} \end{aligned} \quad (17)$$

The basic equations of the theory of chiral plates consist of the equations of motion (7) and (9), the equations of the energy (11) and (13), the constitutive equations (17) and the geometrical equations (16). The field equations can be expressed in terms of the

functions $w_k, \Phi_k, v_\alpha, T_1$ and T_2 . We obtain the following equations

$$\begin{aligned}
& (\mu + \kappa)\Delta w_\alpha + (\lambda + \mu)w_{\rho,\rho\alpha} + C_3\Delta\Phi_\alpha + (C_1 + C_2)\Phi_{\rho,\rho\alpha} \\
& + \kappa\varepsilon_{3\alpha\beta}\Phi_{3,\beta} - b_1T_{1,\alpha} + F_\alpha = \rho\ddot{w}_\alpha \\
& (\mu + \kappa)\Delta w_3 + C_3\Delta\Phi_3 + \kappa\varepsilon_{3\alpha\beta}\Phi_{\beta,\alpha} + \mu v_{\rho,\rho} + F_3 = \rho\ddot{w}_3 \\
& C_3\Delta w_\alpha + (C_1 + C_2)w_{\rho,\rho\alpha} + \gamma\Delta\Phi_\alpha + (\alpha + \beta)\Phi_{\rho,\rho\alpha} + \kappa\varepsilon_{3\alpha\rho}(w_{3,\rho} - v_\rho) \\
& + 2(C_3 - C_2)\varepsilon_{3\alpha\beta}\Phi_{3,\beta} - 2\kappa\Phi_\alpha - b_2T_{1,\alpha} + G_\alpha = J\ddot{\Phi}_\alpha \\
& C_3\Delta w_3 + \gamma\Delta\Phi_3 + \kappa\varepsilon_{3\rho\alpha}w_{\alpha,\rho} + 2\varepsilon_{3\alpha\beta}(C_3 - C_2)\Phi_{\beta,\alpha} \\
& + C_2v_{\alpha,\alpha} - 2\kappa\Phi_3 + G_3 = J\ddot{\Phi}_3 \tag{18} \\
& I[(\mu + \kappa)\Delta v_\alpha + (\lambda + \mu)v_{\rho,\rho\alpha} - b_1T_{2,\alpha}] \\
& - 2h[\mu w_{3,\alpha} + C_2\Phi_{3,\alpha} + \kappa\varepsilon_{3\beta\alpha}\Phi_\beta + (\mu + \kappa)v_\alpha] + H_\alpha = \rho I\ddot{v}_\alpha \\
& k\Delta T_1 - c\dot{T}_1 - T_0b_1\dot{w}_{\alpha,\alpha} - T_0b_2\dot{\Phi}_{\alpha,\alpha} = -S_1 \\
& I[k\Delta T_2 - c\dot{T}_2 - T_0b_1\dot{v}_{\alpha,\alpha}] - 2hkT_2 = -S_2
\end{aligned}$$

where Δ is the two-dimensional Laplacian, and we have used the notation $c = aT_0$. We note that in the case of the centrosymmetric solids the coefficients C_k and b_2 are equal to zero. In this case the system (18) reduces to two uncoupled systems: one for the functions w_α, Φ_3 and T_1 , and the other for the functions $w_3, v_\alpha, \Phi_\alpha$ and T_2 .

To the field equations we must adjoin initial conditions and boundary conditions.

The initial conditions are

$$\begin{aligned}
w_j(x_1, x_2, 0) &= w_j^0(x_1, x_2) \quad \Phi_j(x_1, x_2, 0) = \Phi_j^0(x_1, x_2) \\
v_\alpha(x_1, x_2, 0) &= v_\alpha^0(x_1, x_2) \quad T_\alpha(x_1, x_2, 0) = T_\alpha^0(x_1, x_2) \\
\dot{w}_j(x_1, x_2, 0) &= \vartheta_j^0(x_1, x_2) \quad \dot{\Phi}_j(x_1, x_2, 0) = \omega_j^0(x_1, x_2) \\
\dot{v}_\alpha(x_1, x_2, 0) &= \eta_\alpha^0(x_1, x_2) \quad (x_1, x_2) \in \Sigma
\end{aligned} \tag{19}$$

where the functions $w_j^0, \Phi_j^0, v_\alpha^0, T_\alpha^0, \vartheta_j^0, \omega_j^0$ and η_α^0 are given. The Neumann problem is characterized by the following boundary conditions

$$\begin{aligned}
\tau_{\beta j} n_\beta &= \tilde{\tau}_j \quad \mu_{\beta j} n_\beta = \tilde{\mu}_j \quad M_{\beta\alpha} n_\beta = \tilde{M}_\alpha \\
\chi_\alpha n_\alpha &= \tilde{\chi} \quad Q_\alpha n_\alpha = \tilde{Q} \quad \text{on } \Gamma \times \mathcal{T}
\end{aligned} \tag{20}$$

where the functions $\tilde{\tau}_j, \tilde{\mu}_j, \tilde{M}_\alpha, \tilde{\chi}$ and \tilde{Q} are prescribed. In the case of Dirichlet problem the boundary conditions are

$$w_j = \tilde{w}_j \quad \Phi_j = \tilde{\Phi}_j \quad v_\alpha = \tilde{v}_\alpha \quad T_\alpha = \tilde{T}_\alpha \quad \text{on } \Gamma \times \mathcal{T} \tag{21}$$

where the functions $\tilde{w}_j, \tilde{\Phi}_j, \tilde{v}_\alpha$ and \tilde{T}_α are given.

RECIPROCITY AND UNIQUENESS RESULTS

In this section we present reciprocity and uniqueness theorems in the framework of the dynamic theory. Let F and G be scalar fields on $\Sigma \times \mathcal{T}$ that are continuous in time.

We denote by $F * G$ the convolution of F and G , i.e.

$$F * G(x, t) = \int_0^t F(x, t - \tau) G(x, \tau) d\tau \quad x \in \Sigma \quad t \in \mathcal{T}$$

In what follows we suppose that $\mathcal{T} = (0, \infty)$. Let f and g be functions on \mathcal{T} defined by

$$f(t) = 1 \quad g(t) = t \quad t \in \mathcal{T} \quad (22)$$

If F is a continuous function on $\Sigma \times \mathcal{T}$, then we write \widehat{F} for $f * F$, that is

$$\widehat{F}(x, t) = \int_0^t F(x, \tau) d\tau \quad x \in \Sigma \quad t \in \mathcal{T} \quad (23)$$

We define the functions W_1 and W_2 on $\Sigma \times \mathcal{T}$ by

$$W_1 = \widehat{S}_1 + \rho T_0 \zeta^0 \quad W_2 = \widehat{S}_2 + \rho T_0 \sigma^0 \quad (24)$$

where

$$\rho \zeta^0 = b_1 w_{\alpha, \alpha}^0 + b_2 \Phi_{\alpha, \alpha}^0 + a T_1^0 \rho \sigma^0 = I(b_1 v_{\alpha, \alpha}^0 + a T_2^0) \quad (25)$$

In view of (11), (13), (17), (24) and (25) we obtain

Lemma 1. *The functions $\zeta, \sigma \in C^{0,1}$, $\chi_\alpha, Q_\alpha \in C^{1,0}$ and $\chi_3 \in C^{0,0}$ satisfy Eqs. (11) and (13), and the initial conditions $\zeta(x, 0) = \zeta^0(x)$, $\sigma(x, 0) = \sigma^0(x)$, $x \in \Sigma$, if and only if*

$$\rho T_0 \zeta = \widehat{\chi}_{\alpha, \alpha} + W_1 \quad \rho T_0 \sigma = \widehat{Q}_{\alpha, \alpha} - 2h\widehat{\chi}_3 + W_2 \quad \text{on } \Sigma \times [0, \infty) \quad (26)$$

The proof is immediate.

We consider two data systems of loading for the Neumann problem

$$\begin{aligned} \mathcal{L}^{(\alpha)} = \{ & F_k^{(\alpha)}, G_k^{(\alpha)}, H_\beta^{(\alpha)}, S_\beta^{(\alpha)}, w_k^{0(\alpha)}, \Phi_k^{0(\alpha)}, T_\beta^{0(\alpha)}, v_\rho^{0(\alpha)} \\ & \vartheta_j^{0(\alpha)}, \omega_j^{0(\alpha)}, \eta_\beta^{0(\alpha)}, \widetilde{\tau}_j^{(\alpha)}, \widetilde{\mu}_j^{(\alpha)}, \widetilde{M}_\rho^{(\alpha)}, \widetilde{\chi}^{(\alpha)}, \widetilde{Q}^{(\alpha)} \} \quad (\alpha = 1, 2) \end{aligned}$$

and denote by $A^{(\alpha)} = \{w_k^{(\alpha)}, \Phi_k^{(\alpha)}, v_\beta^{(\alpha)}, T_\beta^{(\alpha)}, \gamma_{ij}^{(\alpha)}, \eta_{\beta k}^{(\alpha)}, \xi_{\beta\alpha}^{(\alpha)}, \zeta^{(\alpha)}, \sigma^{(\alpha)}, \tau_{ij}^{(\alpha)}, \mu_{\beta j}^{(\alpha)}, M_{\kappa\nu}^{(\alpha)}, \chi_k^{(\alpha)}, Q_\beta^{(\alpha)}\}$, a solution corresponding to $\mathcal{L}^{(\alpha)}$. We define the following functions corresponding to the solution $A^{(\alpha)}$,

$$\begin{aligned}\tau_k^{(\alpha)} &= \tau_{\beta k}^{(\alpha)} n_\beta \mu_k^{(\alpha)} = \mu_{\beta k}^{(\alpha)} n_\beta M_\rho^{(\alpha)} = M_{\beta\rho}^{(\alpha)} n_\beta \\ \chi^{(\alpha)} &= \chi_\rho^{(\alpha)} n_\rho Q^{(\alpha)} = Q_\rho^{(\alpha)} n_\rho \\ W_1^{(\alpha)} &= \widehat{S}_1^{(\alpha)} + \rho T_0 \zeta^{0(\alpha)} \quad W_2^{(\alpha)} = \widehat{S}_2^{(\alpha)} + \rho T_0 \sigma^{0(\alpha)}\end{aligned}\tag{27}$$

We introduce the notations

$$\begin{aligned}\Pi_{\kappa\nu}(r, s) &= \int_\Gamma \{2h[\tau_j^{(\kappa)}(r)w_j^{(\nu)}(s) + \mu_j^{(\kappa)}(r)\Phi_j^{(\nu)}(s) - \frac{1}{T_0}\widehat{\chi}^{(\kappa)}(r)T_1^{(\nu)}(s)] \\ &\quad + M_\beta^{(\kappa)}(r)v_\beta^{(\nu)}(s) - \frac{1}{T_0}\widehat{Q}^{(\kappa)}(r)T_2^{(\nu)}(s)]dl \\ &\quad + \int_\Sigma \{2h[F_j^{(\kappa)}(r)w_j^{(\nu)}(s) + G_j^{(\kappa)}(r)\Phi_j^{(\nu)}(s) - \frac{1}{T_0}W_1^{(\kappa)}(r)T_1^{(\nu)}(s)] \\ &\quad + H_\alpha^{(\kappa)}(r)v_\alpha^{(\nu)}(s) - \frac{1}{T_0}W_2^{(\kappa)}(r)T_2^{(\nu)}(s)]da \\ K_{\kappa\nu}(r, s) &= \int_\Sigma \{2h\rho\ddot{w}_j^{(\kappa)}(r)w_j^{(\nu)}(s) + 2hJ\ddot{\Phi}_j^{(\kappa)}(r)\Phi_j^{(\nu)}(s) \\ &\quad + \rho I\ddot{v}_\alpha^{(\kappa)}(r)v_\alpha^{(\nu)}(s) - \frac{k}{T_0}[2h\widehat{T}_{1,\alpha}^{(\kappa)}(r)T_{1,\alpha}^{(\nu)}(s) \\ &\quad + I\widehat{T}_{2,\alpha}^{(\kappa)}(r)T_{2,\alpha}^{(\nu)}(s) + 2h\widehat{T}_2^{(\kappa)}(r)T_2^{(\nu)}(s)]\}da\end{aligned}\tag{28}$$

for all $r, s \in \mathcal{T}$. Here, for convenience, we have suppressed the argument \mathbf{x} .

First we present a reciprocity relation which involves two processes at different instants.

Theorem 1. *Let*

$$E_{\alpha\beta}(r, s) = \Pi_{\alpha\beta}(r, s) - K_{\alpha\beta}(r, s)\tag{29}$$

for all $r, s \in \mathcal{T}$. Then

$$E_{\alpha\beta}(r, s) = E_{\beta\alpha}(s, r) \quad (\alpha, \beta = 1, 2) \quad (30)$$

Proof. We denote

$$\begin{aligned} W_{\kappa\nu}(r, s) = & 2h[\tau_{\beta j}^{(\kappa)}(r)\gamma_{\beta j}^{(\nu)}(s) + \tau_{3\alpha}^{(\kappa)}(r)\gamma_{3\alpha}^{(\nu)}(s) + \mu_{\beta j}^{(\kappa)}(r)\eta_{\beta j}^{(\nu)}(s) \\ & - \rho\zeta^{(\kappa)}(r)T_1^{(\nu)}(s)] + M_{\alpha\beta}^{(\kappa)}(r)\xi_{\alpha\beta}^{(\nu)}(s) - \rho\sigma^{(\kappa)}(r)T_2^{(\nu)}(s) \end{aligned} \quad (31)$$

In view of the constitutive equations (17) we find that

$$W_{\kappa\nu}(r, s) = 2hW_{\kappa\nu}^{(1)}(r, s) + IW_{\kappa\nu}^{(2)}(r, s) \quad (32)$$

where

$$\begin{aligned} W_{\kappa\nu}^{(1)}(r, s) = & \lambda\gamma_{\rho\rho}^{(\kappa)}(r)\gamma_{\eta\eta}^{(\nu)}(s) + (\mu + \kappa)[\gamma_{\alpha j}^{(\kappa)}(r)\gamma_{\alpha j}^{(\nu)}(s) + \gamma_{3\alpha}^{(\kappa)}(r)\gamma_{3\alpha}^{(\nu)}(s)] \\ & + \mu[\gamma_{\beta j}^{(\kappa)}(r)\gamma_{j\beta}^{(\nu)}(s) + \gamma_{3\alpha}^{(\kappa)}(r)\gamma_{\alpha 3}^{(\nu)}(s)] + C_1[\eta_{\rho\rho}^{(\kappa)}(r)\gamma_{\beta\beta}^{(\nu)}(s) + \gamma_{\rho\rho}^{(\kappa)}(r)\eta_{\beta\beta}^{(\nu)}(s)] \\ & + C_2[\eta_{\alpha j}^{(\kappa)}(r)\gamma_{j\alpha}^{(\nu)}(s) + \gamma_{j\beta}^{(\kappa)}(r)\eta_{\beta j}^{(\nu)}(s)] \\ & + C_3[\eta_{\alpha j}^{(\kappa)}(r)\gamma_{\alpha j}^{(\nu)}(s) + \gamma_{\alpha j}^{(\kappa)}(r)\eta_{\alpha j}^{(\nu)}(s)] \\ & + \alpha\eta_{\rho\rho}^{(\kappa)}(r)\eta_{\beta\beta}^{(\nu)}(s) + \beta\eta_{\alpha}^{(\kappa)}(r)\eta_{\alpha}^{(\nu)}(s) \\ & + \gamma\eta_{\alpha j}^{(\kappa)}(r)\eta_{\alpha j}^{(\nu)}(s) - b_1[T_1^{(\kappa)}(r)\gamma_{\rho\rho}^{(\nu)}(s) + T_1^{(\nu)}(s)\gamma_{\rho\rho}^{(\kappa)}(r)] \\ & - b_2[T_1^{(\kappa)}(r)\eta_{\rho\rho}^{(\nu)}(s) + T_1^{(\nu)}(s)\eta_{\rho\rho}^{(\kappa)}(r)] - aT_1^{(\kappa)}(r)T_1^{(\nu)}(s) \end{aligned} \quad (33)$$

and

$$\begin{aligned}
W_{\kappa\nu}^{(2)}(r, s) = & \lambda \xi_{\rho\rho}^{(\kappa)}(r) \xi_{\alpha\alpha}^{(\nu)}(s) + (\mu + \kappa) \xi_{\alpha\beta}^{(\kappa)}(r) \xi_{\alpha\beta}^{(\nu)}(s) \\
& + \mu \xi_{\beta\alpha}^{(\kappa)}(r) \xi_{\alpha\beta}^{(\nu)}(s) - b_1 [T_2^{(\kappa)}(r) \xi_{\rho\rho}^{(\nu)}(s) \\
& + T_2^{(\nu)}(s) \xi_{\rho\rho}^{(\kappa)}(r)] - a T_2^{(\kappa)}(r) T_2^{(\nu)}(s)
\end{aligned} \tag{34}$$

From (32)-(34) we get

$$W_{\kappa\nu}(r, s) = W_{\nu\kappa}(s, r) \tag{35}$$

On the other hand, if we use the equations (7), (9), (26), (17) and (26) we find that

$$\begin{aligned}
W_{\kappa\nu}(r, s) = & 2h[F_k^{(\kappa)}(r)w_k^{(\nu)}(s) + G_k^{(\kappa)}(r)\Phi_k^{(\nu)}(s) - \frac{1}{T_0}W_1^{(\kappa)}(r)T_1^{(\nu)}(s)] \\
& + H_\alpha^{(\kappa)}(r)v_\alpha^{(\nu)}(s) - \frac{1}{T_0}W_2^{(\kappa)}(r)T_2^{(\nu)}(s) - 2h[\rho\ddot{w}_k^{(\kappa)}(r)w_k^{(\nu)}(s) \\
& + J\ddot{\Phi}_k^{(\kappa)}(r)\Phi_k^{(\nu)}(s)] - \rho I\ddot{v}_\alpha^{(\kappa)}(r)v_\alpha^{(\nu)}(s) \\
& + \{2h[\tau_{\beta k}^{(\kappa)}(r)w_k^{(\nu)}(s) + \mu_{\beta k}^{(\kappa)}(r)\Phi_k^{(\nu)}(s) - \frac{1}{T_0}\hat{\chi}_\beta^{(\kappa)}(r)T_1^{(\nu)}(s)] \\
& + M_{\beta\alpha}^{(\kappa)}(r)v_\alpha^{(\nu)}(s) - \frac{1}{T_0}\hat{Q}_\beta^{(\kappa)}(r)T_2^{(\nu)}(s)]\}_{,\beta} \\
& + \frac{k}{T_0}[2h\hat{T}_{1,\alpha}^{(\kappa)}(r)T_{1,\alpha}^{(\nu)}(s) + I\hat{T}_{2,\alpha}^{(\kappa)}(r)T_{2,\alpha}^{(\nu)}(s) + 2h\hat{T}_2^{(\kappa)}(r)T_2^{(\nu)}(s)]
\end{aligned} \tag{36}$$

If we integrate (36) over Σ and use (27)-(29) and the divergence theorem, then we obtain

$$\int_{\Sigma} W_{\kappa\nu}(r, s) da = E_{\kappa\nu}(r, s) \tag{37}$$

From (35) and (37) we obtain the desired result. We introduce the notations

$$\tau_k = \tau_{\alpha k} n_\alpha \quad \mu_k = \mu_{\alpha k} n_\alpha \quad M_\alpha = M_{\beta\alpha} n_\beta \quad \chi = \chi_\alpha n_\alpha \quad Q = q_\alpha n_\alpha \tag{38}$$

Theorem 2. Let $A = \{w_k, \Phi_k, v_\beta, T_\beta, \gamma_{ij}, \eta_{\beta k}, \xi_{\beta\alpha}, \zeta, \sigma, \tau_{ij}, \mu_{\alpha j}, M_{\alpha\beta}, \chi_j, Q_\alpha\}$ be a solution corresponding to the system of loading $\{F_i, G_i, H_\alpha, S_\alpha, w_i^0, \Phi_i^0, T_\alpha^0, v_\alpha^0, \vartheta_j^0, \omega_j^0, \eta_\alpha^0, \tilde{\tau}_j, \tilde{\mu}_j, \tilde{M}_\alpha, \tilde{\chi}, \tilde{Q}\}$ and let

$$\begin{aligned} \Pi(r, s) = & \int_{\Sigma} [2hF_j(r)w_j(s) + 2hG_j(r)\Phi_j(s) - 2hT_0^{-1}W_1(r)T_1(s) \\ & + H_\alpha(r)v_\alpha(s) - T_0^{-1}W_2(r)T_2(s)]da \\ & + \int_L [2h\tau_j(r)w_j(s) + 2h\mu_j(r)\Phi_j(s) - 2hT_0^{-1}\hat{\chi}(r)T_1(s) \\ & + M_\beta(r)v_\beta(s) - T_0^{-1}\hat{Q}(r)T_2(s)]dl \end{aligned} \quad (39)$$

for all $r, s \in \mathcal{T}$. Then

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_{\Sigma} (2h\rho w_j w_j + 2hJ\Phi_j \Phi_j + \rho I v_\alpha v_\alpha) da \right. \\ & \left. + kT_0^{-1} \int_0^t \int_{\Sigma} (2h\hat{T}_{1,\alpha}\hat{T}_{1,\alpha} + I\hat{T}_{2,\alpha}\hat{T}_{2,\alpha} + 2h\hat{T}_2^2) dt da \right\} \\ & = \int_0^t [\Pi(t-s, t+s) - \Pi(t+s, t-s)] ds \\ & + \int_{\Sigma} \{ 2h\rho [\dot{w}_j(0)w_j(2t) + \dot{w}_j(2t)w_j(0)] + 2hJ[\dot{\Phi}_j(0)\Phi_j(2t) + \dot{\Phi}_j(2t)\Phi_j(0)] \\ & + \rho I [\dot{v}_\alpha(0)v_\alpha(2t) + \dot{v}_\alpha(2t)v_\alpha(0)] \} da \end{aligned} \quad (40)$$

Proof. In view of (30) we obtain

$$\int_0^t E_{11}(t+s, t-s) ds = \int_0^t E_{11}(t-s, t+s) ds \quad (41)$$

Let us apply this relation to the solution $A^{(1)} = A$. From (28), (29) and (39) we obtain

$$\begin{aligned}
\int_0^t E_{11}(t+s, t-s)ds &= \int_0^t \Pi(t+s, t-s)ds - \int_0^t \int_{\Sigma} \{2h\rho\ddot{w}_j(t+s)w_j(t-s) \\
&+ 2hJ\ddot{\Phi}_j(t+s)\Phi_j(t-s) + \rho I\ddot{v}_\alpha(t+s)v_\alpha(t-s) \\
&- kT_0^{-1}[2h\widehat{T}_{1,\alpha}(t+s)T_{1,\alpha}(t-s) \\
&+ I\widehat{T}_{2,\alpha}(t+s)T_{2,\alpha}(t-s) + 2h\widehat{T}_2(t+s)T_2(t-s)]\}ds da
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
\int_0^t E_{11}(t-s, t+s)ds &= \int_0^t \Pi(t-s, t+s)ds - \int_0^t \int_{\Sigma} \{2h\rho\ddot{w}_j(t-s)w_j(t+s) \\
&+ 2hJ\ddot{\Phi}_j(t-s)\Phi_j(t+s) + \rho I\ddot{v}_\alpha(t-s)v_\alpha(t+s) \\
&- kT_0^{-1}[2h\widehat{T}_{1,\alpha}(t-s)T_{1,\alpha}(t+s) + I\widehat{T}_{2,\alpha}(t-s)T_{2,\alpha}(t+s) \\
&+ 2h\widehat{T}_2(t-s)T_2(t+s)]\}ds da
\end{aligned} \tag{43}$$

Clearly, if p and q are functions of class C^2 on $[0, \infty)$, then we have

$$\begin{aligned}
\int_0^t p(t+s)\dot{q}(t-s)ds &= -q(0)p(2t) + p(t)q(t) + \int_0^t \dot{p}(t+s)q(t-s)ds \\
\int_0^t \ddot{p}(t+s)q(t-s)ds &= \dot{p}(2t)q(0) - \dot{p}(t)q(t) + \int_0^t \dot{q}(t-s)\dot{p}(t+s)ds \\
\int_0^t \ddot{q}(t-s)p(t+s)ds &= \dot{q}(t)p(t) - \dot{q}(0)p(2t) + \int_0^t \dot{q}(t-s)\dot{p}(t+s)ds
\end{aligned}$$

If we use these relations, then we can present (42) and (43) in a different form. Thus,

the relation (42) can be expressed as

$$\begin{aligned}
\int_0^t E_{11}(t+s, t-s) ds &= \int_0^t \Pi(t+s, t-s) ds - \int_{\Sigma} \{2h\rho[\dot{w}_j(2t)w_j(0) \\
&- \dot{w}_j(t)w_j(t) + \int_0^t \dot{w}_j(t+s)\dot{w}_j(t-s) ds] + 2hJ[\dot{\Phi}_j(2t)\Phi_j(0) \\
&- \dot{\Phi}_j(t)\Phi_j(t) + \int_0^t \dot{\Phi}_j(t+s)\dot{\Phi}_j(t-s) ds] \\
&+ \rho I[\dot{v}_\alpha(2t)v_\alpha(0) - \dot{v}_\alpha(t)v_\alpha(t) + \int_0^t \dot{v}_\alpha(t+s)\dot{v}_\alpha(t-s) ds] \\
&- 2hkT_0^{-1}[\widehat{T}_{1,\alpha}\widehat{T}_{1,\alpha} + \int_0^t T_{1,\alpha}(t+s)\widehat{T}_{1,\alpha}(t-s) ds] \\
&- IkT_0^{-1}[\widehat{T}_{2,\alpha}\widehat{T}_{2,\alpha} + \int_0^t T_{2,\alpha}(t+s)\widehat{T}_{2,\alpha}(t-s) ds] \\
&- 2hkT_0^{-1}[\widehat{T}_2^2 + \int_0^t T_2(t+s)\widehat{T}_2(t-s) ds]\} da
\end{aligned} \tag{44}$$

The relation (43) can be transformed in a similar way. By using (41), (43) and (44) we obtain (40). \square

Theorem 2 forms the basis of the following uniqueness result.

Theorem 3. *Assume that ρ, J and k are strictly positive and a is different from zero. Then the boundary-initial-value problem has at most one solution.*

Proof. If there are two solutions, then their difference A corresponds to null data. In view of (40) and the initial conditions we get

$$\begin{aligned}
&\int_{\Sigma} (2h\rho w_j w_j + 2hJ\Phi_j \Phi_j + \rho I v_\alpha v_\alpha) da \\
&+ \frac{k}{T_0} \int_0^t \int_{\Sigma} (2h\widehat{T}_{1,\alpha}\widehat{T}_{1,\alpha} + I\widehat{T}_{2,\alpha}\widehat{T}_{2,\alpha} + 2h\widehat{T}_2^2) dt da = 0
\end{aligned} \tag{45}$$

By the hypotheses of theorem and (45) we find that

$$w_i = 0 \quad \Phi_i = 0 \quad v_\alpha = 0 \quad \widehat{T}_{1,\alpha} = 0 \quad \widehat{T}_2 = 0 \quad \text{on } \Sigma \times [0, \infty) \quad (46)$$

From (46) we obtain $T_{1,\alpha} = 0$ on $\Sigma \times \mathcal{T}$ so that $\chi_\alpha = 0$ on $\Sigma \times \mathcal{T}$ and $T_1(x, t) = y(t)$, $x \in \Sigma, t \in [0, \infty)$. The constitutive equations (17) and (46) imply that $\rho\zeta = ay$. The energy equation (11) implies that $aT_0\dot{y} = 0$. Since a and T_0 are different from zero we conclude that $y = \text{constant}$ on $\Sigma \times \mathcal{T}$. The initial conditions imply that $y(0) = 0$ so that $T_1 = 0$ on $\Sigma \times I$. From (46) we get also that $T_2 = 0$. The proof is complete. \square

The method to obtain the uniqueness result has been given by Brun [18].

With the help of Theorem 1 we obtain the following reciprocity theorem.

Theorem 4. *Let $A^{(\alpha)}$ be a solution corresponding to the external data system $\mathcal{L}^{(\alpha)}$, ($\alpha = 1, 2$). Then*

$$\begin{aligned} & \int_{\Sigma} [2h\mathcal{F}_i^{(1)} * w_i^{(2)} + 2h\mathcal{G}_i^{(1)} * \Phi_i^{(2)} + \mathcal{H}_\alpha^{(1)} * v_\alpha^{(2)} \\ & - 2hT_0^{-1}g * W_1^{(1)} * T_1^{(2)} - T_0^{-1}g * W_2^{(1)} * T_2^{(2)}] da \\ & + \int_{\Gamma} g * [2h\tau_j^{(1)} * w_j^{(2)} + 2h\mu_j^{(1)} * \Phi_j^{(2)} - 2hT_0^{-1}f * \chi^{(1)} * T_1^{(2)} + M_\beta^{(1)} * v_\beta^{(2)} \\ & - T_0^{-1}f * Q^{(1)} * T_2^{(2)}] dl = \int_{\Sigma} [2h\mathcal{F}_i^{(2)} * w_i^{(1)} + 2h\mathcal{G}_i^{(2)} * \Phi_i^{(1)} \\ & + \mathcal{H}_\alpha^{(2)} * v_\alpha^{(1)} - 2hT_0^{-1}g * W_1^{(2)} * T_1^{(1)} - T_0^{-1}g * W_2^{(2)} * T_2^{(1)}] da \\ & + \int_{\Gamma} g * [2h\tau_j^{(2)} * w_j^{(1)} + 2h\mu_j^{(2)} * \Phi_j^{(1)} - 2hT_0^{-1}f * \chi^{(2)} * T_1^{(1)} \\ & + M_\beta^{(2)} * v_\beta^{(1)} - T_0^{-1}f * Q^{(2)} * T_2^{(1)}] dl \end{aligned} \quad (47)$$

where

$$\begin{aligned}\mathcal{F}_i^{(\alpha)} &= g * F_i^{(\alpha)} + \rho(t\vartheta_i^{0(\alpha)} + w_i^{0(\alpha)}) \\ \mathcal{G}_i^{(\alpha)} &= g * G_i^{(\alpha)} + J(t\omega_i^{0(\alpha)} + \Phi_i^{0(\alpha)}) \quad \mathcal{H}_\rho^{(\alpha)} = g * H_\rho^{(\alpha)} + \rho I(t\eta_\rho^{0(\alpha)} + v_\rho^{0(\alpha)})\end{aligned}\quad (48)$$

Proof. Let us take in (30), $r = \tau$ and $s = t - \tau$. If we integrate with respect to τ from 0 to t , then with the aid of (28) and (29) we obtain

$$\begin{aligned}& \int_\Sigma [2hF_j^{(1)} * w_j^{(2)} + 2hG_j^{(1)} * \Phi_j^{(2)} - 2hT_0^{-1}W_1^{(1)} * T_1^{(2)} \\ & + H_\alpha^{(1)} * v_\alpha^{(2)} - T_0^{-1}W_2^{(1)} * T_2^{(2)}] da \\ & \int_\Gamma [2h\tau_j^{(1)} * w_j^{(2)} + 2h\mu_j^{(1)} * \Phi_j^{(2)} - 2hT_0^{-1}\widehat{\chi}^{(1)} * T_1^{(2)} \\ & + M_\beta^{(1)} * v_\beta^{(2)} - T_0^{-1}\widehat{Q}^{(1)} * T_2^{(2)}] dl \\ & - \int_\Sigma \{2h\rho\ddot{w}_j^{(1)} * w_j^{(2)} + 2hJ\ddot{\Phi}_j^{(1)} * \Phi_j^{(2)} + \rho I\ddot{v}_\alpha^{(1)} * v_\alpha^{(2)} \\ & - kT_0^{-1}[2h\widehat{T}_{1,\alpha}^{(1)} * T_{1,\alpha}^{(2)} + I\widehat{T}_{2,\alpha}^{(1)} * T_{2,\alpha}^{(2)} + 2h\widehat{T}_2^{(1)} * T_2^{(2)}]\} da \\ & = \int_\Sigma [2hF_j^{(2)} * w_j^{(1)} + 2hG_j^{(2)} * \Phi_j^{(1)} - 2hT_0^{-1}W_1^{(2)} * T_1^{(1)} \\ & + H_\alpha^{(2)} * v_\alpha^{(1)} - T_0^{-1}W_2^{(2)} * T_2^{(1)}] da \\ & + \int_\Gamma [2h\tau_j^{(2)} * w_j^{(1)} + 2h\mu_j^{(2)} * \Phi_j^{(1)} - 2hT_0^{-1}\widehat{\chi}^{(2)} * T_1^{(1)} \\ & + M_\beta^{(2)} * v_\beta^{(1)} - T_0^{-1}\widehat{Q}^{(2)} * T_2^{(1)}] dl \\ & - \int_\Sigma \{2h\rho\ddot{w}_j^{(2)} * w_j^{(1)} + 2hJ\ddot{\Phi}_j^{(2)} * \Phi_j^{(1)} + \rho I\ddot{v}_\alpha^{(2)} * v_\alpha^{(1)} \\ & - kT_0^{-1}[2h\widehat{T}_{1,\alpha}^{(2)} * T_{1,\alpha}^{(1)} + I\widehat{T}_{2,\alpha}^{(2)} * T_{2,\alpha}^{(1)} + 2h\widehat{T}_2^{(2)} * T_2^{(1)}]\}\end{aligned}\quad (49)$$

We note that

$$g * \ddot{w}_j^{(\alpha)} = w_j^{(\alpha)} - t\vartheta_j^{0(\alpha)} - w_j^{0(\alpha)} \quad (50)$$

If we take the convolution of the relation (49) with g and use (48) and (50), then we obtain (47). \square

The method to obtain this reciprocal theorem has been established in [19,20].

INSTABILITY OF SOLUTIONS

In this section we suppose that the body forces and heat supply are absent and we consider the following boundary conditions

$$w_j = 0 \quad \Phi_j = 0 \quad v_\alpha = 0 \quad T_\alpha = 0 \quad \text{on } \Gamma \times \mathcal{T} \quad (51)$$

We assume that the coefficients ρ, I, c and k are strictly positive.

The aim of this section is to study the instability of solutions to the equations (18) with the initial conditions (19) and the boundary conditions (51). The method used is strongly based on the choice of a special function [21]. Before to introduce this functions we shall establish some preliminaries. If we integrate with respect to the time the last two equations from (18), then we get

$$\begin{aligned} \frac{k}{T_0} \int_0^t \Delta T_1 ds - b_1 w_{\alpha,\alpha} - b_2 \Phi_{\alpha,\alpha} - a T_1 &= -\rho \zeta^0 \\ \frac{kI}{T_0} \int_0^t \Delta T_2 ds - \frac{2hk}{T_0} \int_0^t T_2 ds - b_1 I v_{\alpha,\alpha} - a I T_2 &= -\rho \sigma^0 \end{aligned} \quad (52)$$

where $\rho \zeta^0$ and $\rho \sigma^0$ have been defined in (25). We denote by $P_\alpha(x_1, x_2)$ the functions

which satisfy the equations

$$k\Delta P_1 = \rho\zeta^0 T_0 \quad k(I\Delta P_2 - 2hP_2) = \rho\sigma^0 T_0 \quad \text{on } \Sigma \quad (53)$$

and the conditions

$$P_\alpha = 0 \quad \text{on } \Gamma \quad (54)$$

If we define the functions z_α by

$$z_\alpha = P_\alpha + \int_0^t T_\alpha ds \quad (55)$$

then we find that z_α satisfy the equations

$$\begin{aligned} \frac{k}{T_0} \Delta z_1 - b_1 w_{\alpha,\alpha} - b_2 \Phi_{\alpha,\alpha} - aT_1 &= 0 \\ \frac{k}{T_0} (I\Delta z_2 - 2hz_2) - I(b_1 v_{\alpha,\alpha} + aT_2) &= 0 \end{aligned} \quad (56)$$

By performing calculations similar to those used to prove Theorem 2, we can obtain the following energy identity

$$\begin{aligned} E(t) &= \int_\Sigma (2h\rho\dot{w}_i\dot{w}_i + 2hJ\dot{\Phi}_i\dot{\Phi}_i + \rho I\dot{v}_\alpha\dot{v}_\alpha + 2haT_1^2 + IaT_2^2) da \\ &\quad + 2h \int_\Sigma (\lambda\gamma_{\rho\rho}\gamma_{\eta\eta} + (\mu + \kappa)(\gamma_{\alpha j}\gamma_{\alpha j} + \gamma_{3\alpha}\gamma_{3\alpha})) \\ &\quad + \mu(\gamma_{\beta j}\gamma_{j\beta} + \gamma_{3\alpha}\gamma_{\alpha 3}) + 2C_1\eta_{\alpha\alpha}\gamma_{\beta\beta} + 2C_2\eta_{\alpha j}\gamma_{j\alpha} \\ &\quad + 2C_3\eta_{\alpha j}\gamma_{\alpha j} + \alpha\eta_{\alpha\alpha}\eta_{\beta\beta} + \beta\eta_{\alpha j}\eta_{j\alpha} + \gamma\eta_{\alpha j}\eta_{\alpha j}) da \\ &\quad + I \int_\Sigma (\lambda\xi_{\alpha\alpha}\xi_{\beta\beta} + (\mu + \kappa)\xi_{\alpha\beta}\xi_{\alpha\beta} + \mu\xi_{\alpha\beta}\xi_{\beta\alpha}) da \\ &\quad + \frac{2k}{T_0} \int_0^t \int_\Sigma (2hT_{1,\alpha}T_{1,\alpha} + IT_{2,\alpha}T_{2,\alpha} + 2hT_2^2) da ds = E(0) \end{aligned} \quad (57)$$

We now define the function $F_{h,\omega}$ on \mathcal{T} by

$$\begin{aligned} F_{h,\omega}(t) &= \int_{\Sigma} (2h\rho w_i w_i + 2hJ\Phi_i \Phi_i + \rho I v_{\alpha} v_{\alpha}) da \\ &\quad + \frac{k}{T_0} \int_0^t \int_{\Sigma} (2hz_{1,\alpha} z_{1,\alpha} + Iz_{2,\alpha} z_{2,\alpha} + 2hz_2^2) da ds + h(t+w)^2 \end{aligned} \quad (58)$$

where h and w are two positive constants that will be determined later. From (58) we find that

$$\begin{aligned} \dot{F}_{h,w}(t) &= 2 \int_{\Sigma} (2h\rho w_i \dot{w}_i + 2hJ\Phi_i \dot{\Phi}_i + \rho I v_{\alpha} \dot{v}_{\alpha}) da \\ &\quad + \frac{2k}{T_0} \int_0^t \int_{\Sigma} (2hz_{1,\alpha} \dot{z}_{1,\alpha} + Iz_{2,\alpha} \dot{z}_{2,\alpha} + 2hz_2 \dot{z}_2) da ds \\ &\quad + 2h(t+w) - \frac{k}{T_0} \int_{\Sigma} (2hP_{1,\alpha} P_{1,\alpha} + IP_{2,\alpha} P_{2,\alpha} + 2hP_2^2) da \\ \ddot{F}_{h,w}(t) &= 2 \int_{\Sigma} (2h\rho(w_i \ddot{w}_i + \dot{w}_i \dot{w}_i) + 2hJ(\Phi_i \ddot{\Phi}_i + \dot{\Phi}_i \dot{\Phi}_i) \\ &\quad + \rho I(v_{\alpha} \ddot{v}_{\alpha} + \dot{v}_{\alpha} \dot{v}_{\alpha})) da \\ &\quad + \frac{2k}{T_0} \int_{\Sigma} (2hz_{1,\alpha} \dot{z}_{1,\alpha} + Iz_{2,\alpha} \dot{z}_{2,\alpha} + 2hz_2 \dot{z}_2) da + 2h \end{aligned} \quad (59)$$

By using the basic equations and the divergence theorem, we get

$$\begin{aligned}
\ddot{F}_{h,\omega}(t) = & 2 \int_{\Sigma} (2h\rho\dot{w}_i\dot{w}_i + 2hJ\dot{\Phi}_i\dot{\Phi}_i + \rho I\dot{v}_\alpha\dot{v}_\alpha) da \\
& - 2 \int_{\Sigma} 2h\{\lambda\gamma_{\rho\rho}\gamma_{\eta\eta} + (\mu + \kappa)[\gamma_{\alpha j}\gamma_{\alpha j} + \gamma_{3\alpha}\gamma_{3\alpha}] + \mu[\gamma_{\beta j}\gamma_{j\beta} + \gamma_{3\alpha}\gamma_{\alpha 3}] \\
& + 2C_1\eta_{\rho\rho}\gamma_{\beta\beta} + 2C_2\eta_{\alpha j}\gamma_{j\alpha} + 2C_3\eta_{\alpha j}\gamma_{\alpha j} + \alpha\eta_{\alpha\alpha}\eta_{\beta\beta} \\
& + \beta\eta_{\alpha j}\eta_{j\alpha} + \gamma\eta_{\alpha j}\eta_{\alpha j}\} da \\
& - 2 \int_{\Sigma} I\{\lambda\xi_{\alpha\alpha}\xi_{\beta\beta} + (\mu + \kappa)\xi_{\alpha\beta}\xi_{\alpha\beta} + \mu\xi_{\alpha\beta}\xi_{\beta\alpha}\} da \\
& - 2 \int_{\Sigma} 2h(\frac{k}{T_0}\Delta z_1 - b_1\gamma_{\rho\rho} - b_2\eta_{\rho\rho})\dot{z}_1 da \\
& - 2 \int_{\Sigma} \{\frac{k}{T_0}(I\Delta z_2 - 2hz_2) - b_1I\xi_{\alpha\alpha}\}\dot{z}_2 da + 2h
\end{aligned} \tag{60}$$

In view of the field equations, we find that

$$\begin{aligned}
\ddot{F}_{h,\omega}(t) = & 2 \int_{\Sigma} (2h\rho\dot{w}_i\dot{w}_i + 2hJ\dot{\Phi}_i\dot{\Phi}_i + \rho I\dot{v}_\alpha\dot{v}_\alpha) dl \\
& - 2 \int_{\Sigma} 2h\{\lambda\gamma_{\rho\rho}\gamma_{\eta\eta} + (\mu + \kappa)[\gamma_{\alpha j}\gamma_{\alpha j} + \gamma_{3\alpha}\gamma_{3\alpha}] + \mu[\gamma_{\beta j}\gamma_{j\beta} + \gamma_{3\alpha}\gamma_{\alpha 3}] \\
& + 2C_1\eta_{\rho\rho}\gamma_{\beta\beta} + 2C_2\eta_{\alpha j}\gamma_{j\alpha} + 2C_3\eta_{\alpha j}\gamma_{\alpha j} + \alpha\eta_{\alpha\alpha}\eta_{\beta\beta} \\
& + \beta\eta_{\alpha j}\eta_{j\alpha} + \gamma\eta_{\alpha j}\eta_{\alpha j}\} da \\
& - 2 \int_{\Sigma} I\{\lambda\xi_{\alpha\alpha}\xi_{\beta\beta} + (\mu + \kappa)\xi_{\alpha\beta}\xi_{\alpha\beta} + \mu\xi_{\alpha\beta}\xi_{\beta\alpha}\} da \\
& - 2 \int_{\Sigma} (2haT_1^2 + IaT_2^2) da + 2h
\end{aligned} \tag{61}$$

From (57) and (61) we obtain

$$\begin{aligned}
\ddot{F}_{h,\omega}(t) = & 4 \int_{\Sigma} (2h\dot{w}_i\dot{w}_i + 2hJ\dot{\Phi}_i\dot{\Phi}_i + \rho I\dot{v}_\alpha\dot{v}_\alpha) da \\
& + \frac{4k}{T_0} \int_0^t \int_{\Sigma} (2hT_{1,\alpha}T_{1,\alpha} + IT_{2,\alpha}T_{2,\alpha} + 2hT_2^2) da ds - 2(E(0) - h)
\end{aligned} \tag{62}$$

We introduce the notation

$$\nu = \frac{2k}{T_0} \int_{\Sigma} (2hP_{1,\alpha}P_{1,\alpha} + IP_{2,\alpha}P_{2,\alpha} + 2hP_2^2) da \quad (63)$$

From (58), (59) and (62) we obtain the inequality

$$F_{h,\omega} \ddot{F}_{h,\omega} - \left(\dot{F}_{h,\omega} - \frac{\nu}{2} \right)^2 \geq 2(h + E(0))F_{h,\omega} \quad (64)$$

In the case $E(0) < 0$, we select $h = -E(0)$ and we take ω so large to guarantee that $\dot{F}_{h,\omega}(0) > \nu$. Following [22] we obtain

$$F_{h,\omega}(t) \geq \frac{F_{h,\omega}(0)\dot{F}_{h,\omega}(0)}{\dot{F}_{h,\omega}(0) - \nu} \exp\left(\frac{\dot{F}_{h,\omega}(0) - \nu}{F_{h,\omega}(0)}t\right) - \frac{\nu F_{h,\omega}(0)}{\dot{F}_{h,\omega}(0) - \nu} \quad (65)$$

The inequality (65) gives the exponential growth of the solutions. We have obtained the following result

Theorem 5. *Assume that the coefficients ρ, J, c and k are strictly positive. If $E(0) < 0$, then the solution becomes unbounded in an exponential way.*

AN EXISTENCE RESULT

Throughout this section we use a semigroup approach (see [22]) to derive an existence theorem in the dynamical theory with the boundary conditions (51). We introduce the

notation

$$\begin{aligned}
2W = & 2h[\lambda\gamma_{\rho\rho}\gamma_{\nu\nu} + (\mu + \kappa)(\gamma_{\alpha j}\gamma_{\alpha j} + \gamma_{3\alpha}\gamma_{\alpha 3}) \\
& + \mu(\gamma_{\beta j}\gamma_{j\beta} + \gamma_{3\alpha}\gamma_{\alpha 3}) + 2C_1\eta_{\alpha\alpha}\gamma_{\beta\beta} + 2C_2\eta_{\alpha j}\gamma_{j\alpha} \\
& + 2C_3\eta_{\alpha j}\gamma_{\alpha j} + \alpha\eta_{\rho\rho}\eta_{\nu\nu} + \beta\eta_{\nu j}\eta_{j\nu} + \gamma\eta_{\rho j}\eta_{\rho j}] \\
& + I[\lambda\xi_{\alpha\alpha}\xi_{\beta\beta} + (\mu + \kappa)\xi_{\alpha\beta}\xi_{\alpha\beta} + \mu\xi_{\alpha\beta}\xi_{\beta\alpha}]
\end{aligned} \tag{66}$$

In what follows we assume that

- (i) ρ, J, c and k are strictly positive;
- (ii) there exists a positive constant C such that

$$2W \geq C(\gamma_{\alpha j}\gamma_{\alpha j} + \gamma_{3\alpha}\gamma_{3\alpha} + \eta_{\alpha j}\eta_{\alpha j} + \xi_{\alpha\beta}\xi_{\alpha\beta}) \tag{67}$$

for every $\gamma_{\alpha j}, \eta_{\alpha j}$ and $\xi_{\alpha\beta}$.

Let $\dot{w}_j = z_j$, $\dot{\Phi} = \Psi_j$, $\dot{v}_j = y_\alpha$ and define $\mathcal{Z} = \{\omega = (w_j, z_j, \Phi_j, \Psi_j, v_\alpha, y_\alpha, T_\alpha); w_j, \Phi_j, v_\alpha \in W_0^{1,2}; z_j, \Psi_j, y_\alpha, T_\alpha \in L_2\}$, where $W_0^{1,2}$ and L^2 are the well-known Hilbert

spaces. Let

$$\begin{aligned}
A_i\omega &= z_i, \quad B_\alpha\omega = \frac{1}{\rho}[(\mu + \kappa)\Delta w_\alpha + (\lambda + \mu)w_{\rho,\rho\alpha} + C_3\Delta\Phi_\alpha \\
&\quad + (C_1 + C_2)\Phi_{\rho,\rho\alpha} + \kappa\varepsilon_{3\alpha\beta}\Phi_{3,\beta} - b_1T_{1,\alpha}] \\
B_3\omega &= \frac{1}{\rho}[(\mu + \kappa)\Delta w_3 + C_3\Delta\Phi_3 + \kappa\varepsilon_{3\alpha\beta}\Phi_{\beta,\alpha} + \mu v_{\beta,\beta}] \\
K_i\omega &= \Psi_i, \quad D_\nu\omega = \frac{1}{J}[C_3\Delta w_\nu + (C_1 + C_2)w_{\rho,\rho\nu} \\
&\quad + \gamma\Delta\Phi_\nu + (\alpha + \beta)\Phi_{\rho,\rho\nu} + \kappa\varepsilon_{\nu\rho 3}(w_{3,\rho} - v_\rho) \\
&\quad + 2(C_3 - C_2)\varepsilon_{3\alpha\beta}\Phi_{3,\beta} - 2\kappa\Phi_\alpha - b_2T_{1,\alpha}] \\
D_3\omega &= \frac{1}{J}[C_3\Delta w_3 + \gamma\Delta\Phi_3 + \kappa\varepsilon_{3\rho\alpha}w_{\alpha,\rho} \\
&\quad + 2\varepsilon_{3\alpha\beta}(C_3 - C_2)\Phi_{\beta,\alpha} + C_2v_{\alpha,\alpha} - 2\kappa\Phi_3] \\
E_\alpha\omega &= y_\alpha, \quad \mathcal{F}_\alpha\omega = \frac{1}{\rho I}\{I[(\mu + \kappa)\Delta v_\alpha + (\lambda + \mu)v_{\rho,\rho\alpha} \\
&\quad - b_1T_{2,\alpha}] - 2h[\mu w_{3,\alpha} + C_2\Phi_{3,\alpha} + \kappa\varepsilon_{3\beta\alpha}\Phi_\beta + (\mu + \kappa)v_\alpha]\} \\
G_1\omega &= \frac{1}{c}(k\Delta T_1 - T_0b_1z_{\alpha,\alpha} - T_0b_2\Psi_{\alpha,\alpha}) \\
G_2\omega &= \frac{1}{c}[I(k\Delta T_2 - T_0b_1y_{\alpha,\alpha}) - 2hkT_2]
\end{aligned} \tag{68}$$

We introduce the operator \mathcal{A} on \mathcal{Z} defined by

$$\mathcal{A}\omega = (A_i\omega, B_i\omega, K_i\omega, D_i\omega, E_\alpha\omega, \mathcal{F}_\alpha\omega, G_\alpha\omega) \tag{69}$$

with the domain

$$\mathcal{D} = \{\omega = (w_i, z_i, \Phi_i, \Psi_i, v_\alpha, y_\alpha, T_\alpha) \in \mathcal{Z}; \mathcal{A}\omega \in \mathcal{Z}\} \tag{70}$$

We note that the domain of the operator is dense. The boundary-initial-value problem

characterized by the relations (18), (19) and (51) can be reduced to the following abstract equation in the space \mathcal{Z} ,

$$\frac{d\omega}{dt} = \mathcal{A}\omega + \mathcal{B}, \quad \omega(0) = \omega_0 \quad (71)$$

where the vectors \mathcal{B} and ω_0 are defined by

$$\mathcal{B} = (0, 0, 0, F_i/\rho, 0, 0, 0, G_i/J, 0, 0, H_\alpha/(\rho I), S_\alpha/c) \quad (72)$$

$$\omega_0 = (w_i^0, \vartheta_j^0, \Phi_j^0, \omega_j^0, v_\alpha^0, \eta_\alpha^0, T_\alpha^0)$$

Let $\omega = (w_j, z_j, \Phi_j, \Psi_j, v_\alpha, y_\alpha, T_\alpha)$ and $\omega' = (w'_j, z'_j, \Phi'_j, \Psi'_j, v'_\alpha, y'_\alpha, T'_\alpha)$. We introduce the inner product

$$\begin{aligned} \langle \omega, \omega' \rangle = & \int_{\Sigma} (2h\rho z_i z'_i + 2hJ\Psi_i\Psi'_i + \rho I v_\alpha v'_\alpha \\ & + 2haT_1^2 + IaT_2^2) da + \int_{\Sigma} 2W^* da \end{aligned} \quad (73)$$

where

$$\begin{aligned} W^* = & 2h[\lambda\gamma_{\rho\rho}\gamma'_{\eta\eta} + (\mu + \kappa)(\gamma_{\alpha j}\gamma'_{\alpha j} + \gamma_{3\alpha}\gamma'_{3\alpha}) \\ & + \mu(\gamma_{\beta j}\gamma'_{\beta j} + \gamma_{3\alpha}\gamma'_{3\alpha}) + C_1(\eta_{\alpha\alpha}\gamma'_{\beta\beta} + \eta'_{\alpha\alpha}\gamma_{\beta\beta}) \\ & + C_2(\eta_{\alpha j}\gamma'_{j\alpha} + \eta'_{\alpha j}\gamma_{j\alpha}) + C_3(\eta_{\alpha j}\gamma'_{\alpha j} + \eta_{\alpha j}\gamma'_{\alpha j}) \\ & + \alpha\eta_{\alpha\alpha}\eta'_{\beta\beta} + \beta\eta_{\alpha j}\eta'_{j\alpha} + \gamma\eta_{\alpha j}\eta'_{\alpha j}] + I[\lambda\xi_{\rho\rho}\xi'_{\alpha\alpha} \\ & + (\mu + \kappa)\xi_{\alpha\beta}\xi'_{\alpha\beta} + \mu\xi_{\alpha\beta}\xi'_{\beta\alpha}] \end{aligned} \quad (74)$$

$$\gamma'_{\alpha j} = w'_{j,\alpha} + \varepsilon_{j\alpha k}\Phi'_k, \quad \gamma'_{3\alpha} = v'_\alpha + \varepsilon_{3\beta\alpha}\Phi'_\beta, \quad \eta'_{\alpha k} = \Phi'_{k,\alpha}, \quad \xi'_{\alpha\beta} = v'_{\beta,\alpha}.$$

This product defines the norm

$$\|\omega\|^2 = \int_{\Sigma} (2h\rho z_i z_i + 2hJ\Psi_i\Psi_i + \rho I v_\alpha v_\alpha + 2haT_1^2 + IaT_2^2) da + 2 \int_{\Sigma} W da \quad (75)$$

where W is given by (67). The norm (75) is equivalent to the usual norm in \mathcal{Z} .

Lemma 2. *Assume that the hypotheses (i) and (ii) hold. Then, for every $\omega \in \mathcal{D}$, we have*

$$\langle \mathcal{A}\omega, \omega \rangle \leq 0 \quad (76)$$

Proof. If we use the relations (68), (69), the divergence theorem and the conditions (51) we get

$$\langle \mathcal{A}\omega, \omega \rangle = -\frac{k}{T_0} \int_{\Sigma} (2hT_{1,\alpha}T_{1,\alpha} + IT_{2,\alpha}T_{2,\alpha} + 2hT_2^2) da$$

In view of (i) we obtain the desired result. \square

Lemma 3. *Let $\rho(\mathcal{A})$ be the resolvent of the operator \mathcal{A} . Then $0 \in \rho(\mathcal{A})$.*

Proof. We have to prove that the equation

$$\mathcal{A}\omega = \mathcal{F} \quad (77)$$

has a solution $\omega = (w_i, z_i, \Phi_i, \Psi_i, v_\alpha, y_\alpha, T_\alpha) \in \mathcal{D}$ for any $\mathcal{F} = (f_1, f_2, \dots, f_{18}) \in \mathcal{Z}$. The equation (77) can be written in the form

$$\begin{aligned} z_i &= f_i \quad B_i\omega = f_{3+i} \quad \Psi_j = f_{6+j} \quad D_i\omega = f_{9+i} \\ y_\alpha &= f_{12+\alpha} \quad \mathcal{F}_\alpha\omega = f_{14+\alpha} \quad G_\alpha\omega = f_{16+\alpha} \end{aligned} \quad (78)$$

From (78) we find that $z_j, \Psi_j, y_\alpha \in W_0^{1,2}$. The last two equations from (78) can be

written as

$$k\Delta T_1 = cf_{17} + T_0 b_1 f_{\alpha,\alpha} + T_0 b_2 f_{6+\alpha,\alpha} \quad (79)$$

$$k\Delta T_2 - \frac{2hk}{I}T_2 = \frac{c}{I}f_{18} + T_0 b_1 f_{12+\alpha,\alpha}$$

We note that we can find the functions $T_1, T_2 \in W_0^{1,2}$ which satisfy (79). The system (78) reduces to

$$\mathcal{L}_i u = g_i \mathcal{M}_i u = g_{3+i} \mathcal{H}_\alpha u = g_{6+\alpha} \quad (80)$$

where $u = (w_i, \Phi_i, v_\alpha)$, and $\mathcal{L}_j, \mathcal{M}_j, \mathcal{H}_\alpha$ and g_s are defined by

$$\begin{aligned} \mathcal{L}_\alpha u &= \frac{1}{\rho}[(\mu + \kappa)\Delta w_\alpha + (\lambda + \mu)w_{\rho,\rho\alpha} + C_3\Delta\Phi_\alpha \\ &\quad + (C_1 + C_2)\Phi_{\rho,\rho\alpha} + \kappa\varepsilon_{3\alpha\beta}\Phi_{3,\beta}] \\ \mathcal{L}_3 u &= \frac{1}{\rho}[(\mu + \kappa)\Delta w_3 + C_3\Delta\Phi_3 + \kappa\varepsilon_{3\alpha\beta}\Phi_{\beta,\alpha}] \\ \mathcal{M}_\nu u &= \frac{1}{J}[C_3\Delta w_\nu + (C_1 + C_2)w_{\rho,\rho\nu} + \gamma\Delta\Phi_\nu \\ &\quad + (\alpha + \beta)\Phi_{\rho,\rho\nu} + \kappa\varepsilon_{\nu\rho 3}(w_{3,\rho} - v_\rho) + 2(C_3 - C_2)\varepsilon_{3\alpha\beta}\Phi_{3,\beta} - 2\kappa\Phi_\alpha] \\ \mathcal{M}_3 u &= \frac{1}{J}[C_3\Delta w_3 + \gamma\Delta\Phi_3 + \kappa\varepsilon_{3\rho\alpha}w_{\alpha,\rho} \\ &\quad + 2\varepsilon_{3\alpha\beta}(C_3 - C_2)\Phi_{\beta,\alpha} + C_2v_{\alpha,\alpha} - 2\kappa\Phi_3] \\ \mathcal{H}_\alpha u &= \frac{1}{\rho I}\{I[(\mu + \kappa)\Delta v_\alpha + (\lambda + \mu)v_{\rho,\rho\alpha}] \\ &\quad - 2h[\mu w_{3,\alpha} + C_2\Phi_{3,\alpha} + \kappa\varepsilon_{3\beta\alpha}\Phi_\beta + (\mu + \kappa)v_\alpha]\} \\ g_\alpha &= f_{3+\alpha} + \frac{1}{\rho}b_1T_{1,\alpha}, \quad g_3 = f_6, \quad g_{3+\alpha} = f_{9+\alpha} + \frac{1}{J}b_2T_{1,\alpha} \\ g_6 &= f_{12}, \quad g_{6+\alpha} = f_{14+\alpha} + \frac{1}{\rho I}b_1T_{2,\alpha} \end{aligned} \quad (81)$$

Let $u = (w_i, \Phi_i, v_\alpha)$ and $u^* = (w_i^*, \Phi_i^*, v_\alpha^*)$. To study the system (80) we introduce the bilinear form

$$\Lambda(u, u^*) = \int_{\Sigma} \Pi da$$

where

$$\Pi = 2h[\rho w_j^* \mathcal{L}_j u + J \Phi_j^* \mathcal{M}_j u] + \rho I v_\alpha^* \mathcal{H}_\alpha u$$

By using the divergence theorem and the boundary conditions we can see that Λ is a bounded bilinear form. In view of (67) we find that this form is coercive. On the basis of Lax-Milgram theorem (see, e.g., [23]) we obtain the existence of solution of the system (80). We conclude that the equation (77) has a solution in \mathcal{D} . \square

A direct consequence of the Lumer-Phillips theorem (see [24]) is the following result.

Theorem 6. *The operator \mathcal{A} is the generator of a C^∞ -semigroup of contraction in the Hilbert space \mathcal{Z} .*

Thus we can state the main result of this section

Theorem 7. *Let $\mathcal{B} \in C^1(\mathbb{R}^+, \mathcal{Z}) \cap C^0(\mathbb{R}^+, \mathcal{D})$ and $w_0 \in \mathcal{D}$. Then, there exists a unique solution $\omega \in C^1(\mathbb{R}^+, \mathcal{Z}) \cap C^0(\mathbb{R}^+, \mathcal{D})$ to the dynamic problem.*

DEFORMATION OF A PLATE WITH A CIRCULAR HOLE

In this section we study the deformation of an infinite plate with a circular hole, subjected to a constant temperature at the periphery of the hole. The domain Σ is defined by $\Sigma = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 > b^2, x_3 = 0\}$ where b is a positive constant. We

introduce the notation $r = (x_1^2 + x_2^2)^{1/2}$. We consider the equilibrium theory and assume that

$$F_j = 0 \quad G_j = 0 \quad H_\alpha = 0 \quad S_\alpha = 0 \quad \text{on } \Sigma \quad (82)$$

$$\tilde{\tau}_k = 0 \quad \tilde{\mu}_k = 0 \quad \tilde{M}_\alpha = 0 \quad \tilde{T}_1 = T^* \quad \tilde{T}_2 = 0 \quad \text{on } r = b$$

where T^* is a prescribed constant. In this case the static version of the equations (18) reduces to

$$\begin{aligned} (\mu + \kappa)\Delta w_\alpha + (\lambda + \mu)w_{\rho,\rho\alpha} + C_3\Delta\Phi_\alpha + (C_1 + C_2)\Phi_{\rho,\rho\alpha} + \kappa\varepsilon_{3\alpha\beta}\Phi_{3,\beta} &= b_1T_{1,\alpha} \\ (\mu + \kappa)\Delta w_3 + C_3\Delta\Phi_3 + \kappa\varepsilon_{3\alpha\beta}\Phi_{\beta,\alpha} + \mu v_{\alpha,\alpha} &= 0 \\ C_3\Delta w_\alpha + (C_1 + C_2)w_{\rho,\rho\alpha} + \gamma\Delta\Phi_\alpha + (\alpha + \beta)\Phi_{\rho,\rho\alpha} + \kappa\varepsilon_{3\alpha\rho}(w_{3,\rho} - v_\rho) \\ + 2(C_3 - C_2)\varepsilon_{3\alpha\beta}\Phi_{3,\beta} - 2\kappa\Phi_\alpha &= b_2T_{1,\alpha} \\ C_3\Delta w_3 + \gamma\Delta\Phi_3 + \kappa\varepsilon_{3\rho\alpha}w_{\alpha,\rho} + 2\varepsilon_{3\alpha\beta}(C_3 - C_2)\Phi_{\beta,\alpha} + C_2v_{\alpha,\alpha} - 2\kappa\Phi_3 &= 0 \\ I[(\mu + \kappa)\Delta v_\alpha + (\lambda + \mu)v_{\rho,\rho\alpha}] - 2h[\mu w_{3,\alpha} \\ + C_2\Phi_{3,\alpha} + \kappa\varepsilon_{3\beta\alpha}\Phi_\beta + (\mu + \kappa)v_\alpha] &= Ib_1T_{2,\alpha} \\ \Delta T_1 = 0 \quad \Delta T_2 - \nu^2 T_2 &= 0 \end{aligned} \quad (83)$$

where $\nu = (2h/I)^{1/2}$. It follows from (82) and (83) that the functions T_1 and T_2 are given by

$$T_1 = T^* \quad T_2 = 0 \quad (84)$$

We seek the solution of the equations (83) in the form

$$w_\alpha = F_{,\alpha} \quad w_3 = 0 \quad \Phi_\alpha = G_{,\alpha} \quad \Phi_3 = 0 \quad v_\alpha = \varepsilon_{3\alpha\beta}H_{,\beta} \quad (85)$$

where F, G and H are unknown function of r . It is a simple matter to see that the equations (83) are satisfied if the functions F, G and H satisfy the following equations

$$\begin{aligned}(\lambda + 2\mu + \kappa)\Delta F + (C_1 + C_2 + C_3)\Delta G &= 0 \\(C_1 + C_2 + C_3)\Delta F + (\alpha + \beta + \gamma)(\Delta - p_1^2)G + \kappa H &= 0 \\(\mu + \kappa)(\Delta - \nu^2)H + \kappa\nu^2 G &= 0\end{aligned}\tag{86}$$

where

$$p_1^2 = 2\kappa/(\alpha + \beta + \gamma)$$

The equations (86) implies that the functions G and H satisfy the system

$$(\Delta - p_2^2)G + a_1 H = 0 \quad (\Delta - \nu^2)H + \ell_1 G = 0\tag{87}$$

where we have introduced the notations

$$\begin{aligned}p_2^2 &= 2\kappa(\lambda + 2\mu + \kappa)/d \quad a_1 = \kappa(\lambda + 2\mu + \kappa)/d \\d &= (\lambda + 2\mu + \kappa)(\alpha + \beta + \gamma) - (C_1 + C_2 + C_3)^2 \quad \ell_1 = \kappa\nu^2/(\mu + \kappa)\end{aligned}$$

We consider the representation

$$G = (\Delta - \nu^2)\Lambda \quad H = -\ell_1\Lambda\tag{88}$$

where Λ is a function of class C^4 . Let us denote

$$D = \Delta\Delta - (\nu^2 + p_2^2)\Delta + p_2^2\nu^2 - a_1\ell_1\tag{89}$$

The functions G and H given by (88) satisfy the equations (56) if the function Λ satisfies the equation

$$D\Lambda = 0\tag{90}$$

We can prove this assertion by substituting the functions G and H from (88) into system (87). The operator D can be expressed in the form

$$D = (\Delta - k_1^2)(\Delta - k_2^2)$$

where k_α^2 are the roots of the equation

$$y^2 - (\nu^2 + p_2^2)y + p_2^2\nu^2 - a_1\ell_1 = 0$$

Let assume that k_α are distinct positive constants. We can write

$$\Lambda = A_1 e_1 + A_2 e_2$$

where A_α are arbitrary constants and the functions e_β satisfy the equations

$$(\Delta - k_\alpha^2)e_\alpha = 0 \quad (\text{no sum; } \alpha = 1, 2) \quad (91)$$

The functions e_α that satisfy the equations (91) and vanish at infinity are given by

$$e_\alpha = K_0(k_\alpha r) \quad (\alpha = 1, 2, 3)$$

where K_0 denotes the modified Bessel function of the third kind and zeroth order. Thus, we find that

$$\Lambda = \sum_{\alpha=1}^2 A_\alpha K_0(k_\alpha r) \quad (92)$$

In view of (88) and (92) we obtain

$$G = \sum_{\alpha=1}^2 b_{1\alpha} A_\alpha K_0(k_\alpha r) \quad H = \sum_{\alpha=1}^2 b_{2\alpha} A_\alpha K_0(k_\alpha r) \quad (93)$$

where

$$b_{1\alpha} = k_\alpha^2 - \nu^2 \quad b_{21} = b_{22} = -\ell_1$$

From (86) we find that the function F has the form

$$F = -\vartheta G + A_0 + A_3 \ln r \quad (94)$$

where A_0 and A_3 are arbitrary constants and

$$\vartheta = (C_1 + C_2 + C_3)/(\lambda + 2\mu + \kappa)$$

It follows from (16), (17), (84) and (85) that on the boundary $r = b$ we have

$$\begin{aligned} \tau_{\beta\alpha} n_\beta &= n_\alpha [\lambda \Delta F + (2\mu + \kappa) F'' + C_1 \Delta G + (C_2 + C_3) G'' - b_1 T^*] \\ \mu_{\beta\alpha} n_\beta &= n_\alpha [\alpha \Delta G + (\beta + \gamma) G'' + C_1 \Delta F + (C_2 + C_3) F'' - b_2 T^*] \\ \tau_{\alpha 3} n_\alpha &= 0 \quad \mu_{\alpha 3} n_\alpha = 0 \quad M_{\beta\alpha} n_\beta = \varepsilon_{3\alpha\beta} n_\beta I[(\mu + \kappa) H'' - \mu b^{-1} H'] \end{aligned} \quad (95)$$

where $f' = df/dr$ and $f'' = d^2f/dr^2$. In view of (20), (82) and (95) the boundary conditions on the boundary $r = b$ become

$$\begin{aligned} \lambda \Delta F + (2\mu + \kappa) F'' + C_1 \Delta G + (C_2 + C_3) G'' &= b_1 T^* \\ \alpha \Delta G + (\beta + \gamma) G'' + C_1 \Delta F + (C_2 + C_3) F'' &= b_2 T^* \\ b(\mu + \kappa) H'' - \mu H' &= 0 \quad \text{on } r = b \end{aligned} \quad (96)$$

If we use (93) and (94), then the conditions (96) reduce to

$$\begin{aligned} \sum_{\alpha=1}^2 c_{1\alpha} A_\alpha - (2\mu + \kappa) b^{-2} A_3 &= b_1 T^* \\ \sum_{\alpha=1}^2 c_{2\alpha} A_\alpha - (C_2 + C_3) b^{-2} A_3 &= b_2 T^* \quad \sum_{\alpha=1}^2 c_{3\alpha} A_\alpha = 0 \end{aligned} \quad (97)$$

where

$$\begin{aligned}
c_{1\alpha} &= \{[C_1 + C_2 + C_3 - \vartheta(\lambda + 2\mu + \kappa)]k_\alpha^2 K_0(k_\alpha b) \\
&\quad + [C_2 + C_3 - \vartheta(2\mu + \kappa)]b^{-1}k_\alpha K_1(k_\alpha b)\}b_{1\alpha} \\
c_{2\alpha} &= \{[\alpha + \beta + \gamma - \vartheta(C_1 + C_2 + C_3)]k_\alpha^2 K_0(k_\alpha b) \\
&\quad + [\beta + \gamma - \vartheta(C_2 + C_3)]k_\alpha b^{-1} K_1(k_\alpha b)\}b_{1\alpha} \\
c_{3\alpha} &= bk_\alpha b_{2\alpha}\{(\mu + \kappa)[k_\alpha K_0(k_\alpha b) + b^{-1} K_1(k_\alpha b)] + \mu(k_\alpha b)\}
\end{aligned}$$

Here, K_1 denotes the modified Bessel function of the third kind and order 1. From (97) we determine the constants A_s . The solution of the problem is given by (85), (93) and (94). The behaviour of solution at infinity is the same as in the classical theory. In contrast with the theory of achiral plates, a uniform temperature field acting on the boundary of the hole produces a bending effect.

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